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# THE TRANSVERSE RESISTIVE WALL INSTABILITY OF EXTREMELY RELATIVISTIC BEAM OF ELECTRONS AND POSITRONS

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(Presented by B. Touschek)

1. This note wants to be a summary of the main results on the transverse resistive wall instability, obtained following a line of investigation which was started by the Adone group in 1964 (1).

A detailed description of the work is given in Ref. 2, which in the following will be referred to as F P T.

The procedure followed in this work is:

1) to determine the electromagnetic fields, subject to the boundary conditions on the walls of the vacuum tank, due to a given longitudinal motion and an arbitrary vertical motion of a single particle;

2) integrate the mechanical equations of motion when $N_1$ and $N_2$ particles are present in the two beams and determine the eigen-values of the frequency of the transverse motion and their imaginary parts, which according to their sign, tell us if the motion is stable or unstable.

The actual calculation make use of a number of simplifications. The most important are:

a) synchrotron oscillations are neglected and the longitudinal motion of the particles is assumed to be rectilinear and described by

$$Z_K(t) = \pm vt + \xi_K \quad [1]$$

where Z is the longitudinal coordinate, $K = 1, 2 \ldots N_1$ for electrons and $K = 1, 2 \ldots N_2$ for positrons, $+v$ is the velocity of electrons and positrons respectively, $\xi_k$ is a constant;

b) to adapt the calculation to the case of circular accelerators a periodicity condition is imposed on the rectilinear motion and in the same time the limitation $0 \leq \xi_k < u$ is introduced, u being the circumference of the machine;

c) the vertical motion of the particles in the absence of the wall is assumed to be harmonic, all the oscillators having the same frequency $\nu_0$;

d) the problem is treated in the linear approximation;

e) the effects of the electromagnetic forces obtained in the limit of infinite wall conductivity, and hence all the variations of the real part of the frequency, are not considered.

The last assumption can easily be removed and is used here only in order not to introduce inessential complications.

2. A detailed derivation of the fields in the presence of the resistive wall is given in F P T, where it is shown that the force on the K-th particle due to the i-th particle can be written as

$$\rho\left(\mp\frac{Z_i - Z_K}{v}\right)\eta_i\left(t \pm \frac{Z_i - Z_K}{v}\right) \quad [2]$$

where $\eta_i(t) = \eta_i e^{-i\nu t}$ is the transverse coordinate of the i-th particle and

$$\rho(\tau) = \int d\omega\, P(\omega)\, e^{-i\omega\tau} \quad [3]$$

The function $P(\omega)$, which is derived in FPT, is quite complicated, but can be reasonably approximated by (see F P T)

$$\omega_0 P(\omega) = \frac{(1+i)A}{\sqrt{\omega/\omega_0}} \quad [4]$$

with

$$A = r_0 v c Z_0 / 2\pi\gamma a^3 \quad [5]$$

here $r_0$ is the classical electron radius,

$$Z_0 = \sqrt{\frac{\omega_0}{8\pi\sigma}},$$

$\sigma$ is the conductivity of the wall, $\gamma$ is the electron energy divided by its rest mass, a is the vertical half-dimension of the vacuum tank, and $\omega_0$ is the revolution frequency.

Using $P(\omega)$ defined by [4] it is easy to see that

$$\rho(\tau) = 0 \quad \text{if} \quad \tau < 0$$

and this result, which means that the electromagnetic fields giving the resistive wall force lay only behind the generating particle, allows us to fix the signs in [2] in the different cases of electron-electron, electron-positron, or positron-positron interactions.

Now, using the periodicity condition, i.e. replacing in [2] $Z_i - Z_k$ by $Z_i - Z_k + n u$ and summing over n, the equations of motion for the single beam case are obtained, namely

$$(-\nu^2 + \nu_0^2)\,\eta_K = \omega_o \sum_{i,s} P(\nu + s\,\omega_o)\, e^{-i\frac{\xi_i - \xi_K}{R}s}\,\eta_i \quad [6]$$

where

$$\omega_o = \frac{2\pi v}{u} = \frac{v}{R}$$

and R is the radius of the machine. Introducing the quantities

$$N g_r = \sum_K e^{ir\xi_K/R} \quad [7]$$

which characterize the longitudinal distribution of the electrons, and

$$Y_r = \sum_K \eta_K e^{-ir\xi_K/R} \quad [8]$$

eq. [4] becomes

$$(-\nu^2 + \nu_0^2)\, Y_r = N\,\omega_o \sum P(\nu + s\,\omega_o)\, g_{s-r} Y_s \quad [9]$$

Eq. [9] apply for an arbitrary longitudinal particle distribution.

The determination of the eigenvalues of $\nu$ in this general case is not trivial but it becomes very simple in the two particular cases of a uniform longitudinal distribution and of point like bunches.

For practical purposes the two sums in [7] [8] can be substitued by integrals, after the introduction of the longitudinal distribution function $l(\xi)$.

Then one has

$$N g_r = \int_0^u e^{ir\xi/R}\; l(\xi)\, d\xi \quad [7]$$

$$Y_r = \int_0^u \eta(\xi)\, e^{-ir\xi/R}\; l(\xi)\, d\xi \quad [8]$$

3) We consider here the case of uniform longitudinal distribution, i. e. $l(\xi) = \text{constant}$ and hence

$$g_r = \delta_{r,o}$$

Eq. [8] becomes simply

$$(-\nu^2 + \nu_o^2)\, Y_r = N\,\omega_o\, P(\nu + r\,\omega_o)\, Y_r \quad [10]$$

so that the $Y_r$ are now the normal modes of our problem, with eigenvalues

$$\nu^2 = \nu_o^2 - N\,\omega_o\, P(\nu_o + r\,\omega_o) \quad [11]$$

The function $P(\omega)$ on the r. h. s. of [10] has been evaluated for $\nu = \nu_o$ in order to obtain $\nu_r$. From [10] and [4] it follows, introducing the number $q = \nu_o/\omega_o$ the well known result (3) that $\text{Im}\,\nu_r < 0$ provided that $q + r > 0$ and $\text{Im}\nu_r > 0$ provided that $q + r < 0$. (We have defined

$$\sqrt{-1 \times 1} = i\,\sqrt{1 \times 1} \;(2).$$

In the unstable case $\text{Im}\,\nu_r > 0$, the rise time defined as $1/t_r = 2\,\text{Im}(\nu^2/\nu^0)$ is given by

$$t_r = \frac{\pi a^3}{N r_o c R} Z_o^{-1} q \sqrt{-(q + r)} \quad [12]$$

4. The case of extremely bunched beams having an equal number of particles can be treated by assuming

$$l(\xi) = \frac{N}{B}\sum_{n=1}^{B} \delta(\xi - \xi_n) \equiv \frac{N}{B}\sum_{n=1}^{B} \delta\left(\xi - \frac{n u}{B}\right)$$

where B is the number of bunches.

Then one has

$$g_r = \delta_{r,\,nB} \qquad n = 0, \pm 1, \pm 2, \ldots$$

and [8] becomes

$$(-\nu^2 + \nu_o^2)\, Y_r = \omega_o N \sum_n P(\nu + (r + nB)\,\omega_o)\, Y_r \quad [13]$$

where use has been made of the fact that in this case

$$Y_r = \frac{N}{B}\sum_{n=1}^{B} e^{-i\,2\pi r\, m/B}\,\eta_m = Y_{r+nB}\,,$$

so that we have now only B normal modes.

The imaginary part of the eigenvalues is

$$\text{Im}\,(\nu_r)^2 = -\,\omega_o N\, \text{Im} \sum_n P((q + r + nB)\,\omega_o) =$$

$$= -\frac{N A}{\sqrt{B}}\, T\left(\frac{q + r}{B}\right) \quad [14]$$

where

$$T\left(\frac{q+r}{B}\right) = \sum_{n > -\frac{q+r}{B}} \left(n + \frac{q+r}{B}\right)^{-1/2}$$

$$- \sum_{n > -\frac{q+r}{B}} \left(n - \frac{q+r}{B}\right)^{-1/2} \qquad [15]$$

One can easily see that the function T (x) has the following properties:

$T(x+1) = T(x)$

$T(1-x) = -T(x)$

$T(1/2) = 0$

$T(x) > 0 \qquad \text{for } 0 < x < 1/2$

$T(x) < 0 \qquad \text{for } 1/2 < x < 1$

The values of T (x) for $0 < x < 1/2$ are given in the table

| | | | | | | |
|---|---|---|---|---|---|---|
| x | 0.00 | 0.05 | 0.10 | 0.15 | 0.20 | 0.25 |
| T (x) | 00 | 4.35 | 2.90 | 2.19 | 1.72 | 1.34 |
| x | 0.30 | 0.35 | 0.40 | 0.45 | 0.50 | |
| T (x) | 1.03 | 0.75 | 0.51 | 0.25 | 0.00 | |

We therefore see that the modes such that $0 < (q + r/B) < 1/2$ mod (1) are stable and the modes such that $1/2 < (q + r/B) < 1$ mod (1), are unstable.

This rule is in agreement with the results of Courant and Sessler (4) (5).

5. The last topic we consider is the resistive wall effect for an electron-positron storage ring.

It follows from equation [2] that in the presence of two beams travelling in identical orbits the mechanical equations will be given by

$$(-\nu^2 + \nu_o^2)\,\eta_u^{(2)} = \cdots +$$

$$+ \omega_o \sum_s P[(q+s)\omega_o]\, e^{-is(\xi_K^{(2)} - \xi_i^{(1)})/R - 2 i s\omega_o t}\ \eta_i^{(1)}$$

[16]

Here the indices (1) and (2) label the electron and positron respectively and the dots indicate the interaction of a beam with itself.

The equation for the electron beam is obtained by exchanging the indices (1) and (2).

The mechanical problem posed by [16] is different from what has been dealt with so far: the time dependence does not cancel on the right hand side of [16]. In addition to a possible antidamping the resistive wall force therefore "deharmonizes" the betatron oscillations.

Whe are not interested in the latter part of this effect though it should be pointed out that it might lead to complications, whenever the free betatron oscillations are strongly anharmonic.

We can then consider only the average value i.e. the $S = 0$ term, of the resistive wall force on the right hand side of [16].

These equations then become

$$(-\nu^2 + \nu_o^2)\, Y_r^{(2)} = \omega_o N_2 \sum_s P[(q+s)\omega_o]\, g_{s-r}^{(2)} Y_s^{(2)} +$$

$$+ \omega_o P(q\omega_o) N_2\, g_{-r}^{(2)} Y_o^{(1)} \qquad [17]$$

$$(-\nu^2 + \nu_o^2)\, Y_r^{(1)} = \omega_o N_1 \sum_s P[(q+s)\omega_o]\, g_{s-r}^{(1)} Y_s^{(1)} +$$

$$+ \omega_o P(q\omega_o) N_1\, g_{-r}^{(1)} Y_o^{(2)}$$

Considering again the case of point like bunches [17] gives us

$$(-\nu^2 + \nu_o^2)\, Y_r^{(2)} = \omega_o N_2 \sum_n P[(q + r + nB)\omega_o]\, Y_r^{(2)} +$$

$$+ \omega_o P(q\omega_o) N_2\, \delta_{r,nB}\, Y_o^{(1)} \qquad [18]$$

$$(-\nu^2 + \nu_o^2)\, Y_r^{(1)} = \omega_o N_1 \sum_n P[(q + r + nB)\omega_o]\, Y_r^{(1)} +$$

$$+ \omega_o P(q\omega_o) N_1\, \delta_{r,nB}\, Y_o^{(2)}$$

$$n = 0, \pm 1, \pm 2, \ldots$$

Equation [18] shows that the single beam normal modes with $r \neq 0$ are still normal modes of the coupled problem with the same eigenvalues.

For the $r = 0$ term the normal modes are a mixture of the single beam modes and the eigenvalues are given by

$$2\nu^2 = 2\nu_o^2 - \alpha(N_1 + N_2) \pm \{\alpha^2 (N_1 - N_2)^2 + 4\beta^2 N_1 N_2\}^{1/2} \qquad [19]$$

where

$$\alpha = \omega_o \sum_n P[(q + nB)\omega_o]$$

$$\beta = \omega_o P(q\omega_o)$$

It is clear from [19] that the imaginary part of $\nu$ depends not only on $N_1$, $N_2$ but also on the real parts of $\alpha$, $\beta$.

A remarkable exception is represented by the case $N_1 = N_2 = N$, for which

$$I_m(\nu^2) = -NA\left\{T\left(\frac{q}{B}\right) \pm \frac{1}{\sqrt{q}}\right\} \qquad [20]$$

In this case no additional instability is created by the beam-beam interaction, provided that

$$T\left(\frac{q}{B}\right) > 0, \quad T\left(\frac{q}{B}\right) > \frac{1}{\sqrt{q}}$$

This result can be generalized to arbitrary numbers $N_1$, $N_2$ if special assumptions are made about the coefficients $\alpha$ and $\beta$.

Writing $\alpha = \alpha_1 + i\,\alpha_2$, $\beta = \beta_1 + i\,\beta_2$, if it is assumed that $\beta_1$ is much bigger than the moduli of all the other coefficients ($\beta_2$, $\alpha_1$ and $\alpha_2$) we can suppress the term with $\alpha$ in the square root in [19]. In this case the imaginary part of the r.h.s. of [19] is exclusively determined by the imaginary parts of $\alpha$ and $\beta$ and [20] can be generalized to give

$$\mathrm{Im}(\nu^2) = -A\left(\frac{1}{2}(N_1 + N_2)\,T\,\frac{q}{B} \pm \sqrt{N_1 N_2}\,q^{-1/2}\right) \qquad [21]$$

which clearly goes into [20] for $N_1 = N_2 = N$. Since the geometrical mean, $\sqrt{N_1 N_2}$, is always less than the arithmetical mean, $1/2\,(N_1 + N_2)$, it follows that for an arbitrary number of particles the resistive wall interaction of two beams does not introduce any new instabilities provided that

$$T\left(\frac{q}{B}\right) > 0 \quad \text{and} \quad q^{-1/2} < T\left(\frac{q}{B}\right).$$

This is essentially the result obtained by Pellegrini and Sessler (6).

6. The results obtained show clearly that the machine parameters and in particular the values of q and B, play an important role in defining the beams behaviour with regard to the transverse coherent resistive wall instability.

For instance to avoid the introduction of a new instability term when two beams circulate in the same machine, one must satisfy the condition $T\,(q/B) > 1/\sqrt{q}$ and this is most easily achieved by choosing

$$q = nB + \varepsilon B, \quad 0 < \varepsilon < \frac{1}{2}, \quad n = 0, 1, \ldots$$

This rule means also that for the bunched single beam case the $r = 0$ mode is stable.

Considering the other modes, a certain number of which is always unstable, it can be easily seen that a small pick-up electrode will respond to all the frequences $\omega_0(q + s + nB)$ with $n = 0, \pm 1, \pm 2, \ldots$. It follows that an instability of the s-th mode can be cured by applying a feed-back on any one of these frequencies.

A particularly simple situation is that of $B = 1$; in this case the beams are stable if $n < q < n + 1/2$, where n is any integer, and there is no need applying a r.f. feed-back.

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